

GROUPS WITH NO COARSE EMBEDDINGS INTO HYPERBOLIC GROUPS

DAVID HUME AND ALESSANDRO SISTO

ABSTRACT. We introduce an obstruction for the existence of a coarse embedding of a given group or space into a hyperbolic group, or more generally into a hyperbolic graph of bounded degree. The condition we consider is “admitting exponentially many fat bigons”, and it is preserved by a coarse embedding between graphs with bounded degree. Groups with exponential growth and linear divergence (such as direct products of two groups one of which has exponential growth, solvable groups that are not virtually nilpotent, and uniform higher-rank lattices) have this property and hyperbolic graphs do not, so the former cannot be coarsely embedded into the latter. Other examples include certain lacunary hyperbolic and certain small cancellation groups.

1. INTRODUCTION

Hyperbolic groups have been at the heart of geometric group theory since Gromov’s seminal paper [Gro87] and are still of vital importance to the present day. They are among the best understood classes of groups with a large, diverse and ever-expanding literature. Despite this it is not at all well understood which finitely generated groups may appear as subgroups of hyperbolic groups. One algebraic obstruction is admitting a Baumslag–Solitar subgroup $BS(m, n) = \langle a, b \mid b^{-1}a^mba^{-n} \rangle$ with $|m|, |n| \geq 1$. The goal of this paper is to consider a more geometric obstruction. To do this, we consider every finitely generated group as a Cayley graph with respect to some finite symmetric generating set, and consider every graph as a metric space with the shortest path metric.

In the same way that one may view the existence of a quasi-isometry $q : H \rightarrow G$ between finitely generated groups as the natural geometric generalization of the algebraic statement “ H and G are (abstractly) commensurable”, we will consider the existence of a coarse embedding $\phi : H \rightarrow G$ as the comparable generalization of the statement “ H is virtually isomorphic to a subgroup of G ”. In both cases the algebraic statement is known to be stronger than the geometric one: all Baumslag–Solitar groups $BS(m, n)$ with $1 < |m| < |n|$ are quasi-isometric, but, for example, $BS(2, p)$ and $BS(2, q)$ are not commensurable whenever p, q are distinct odd primes [Why01]; while \mathbb{Z}^2 is never a subgroup of a hyperbolic group, but \mathbb{R}^2 coarsely embeds into real hyperbolic 3-space (as a horosphere) and hence into the fundamental group of any closed hyperbolic 3-manifold.

Coarse embeddings of groups into other spaces (particularly certain Banach spaces) are also highly sought, since groups admitting such an embedding satisfy the Novikov and coarse Baum–Connes conjectures [Yu00].

There are few invariants which can provide a general geometric obstruction to a coarse embedding, of which the most commonly studied are growth and asymptotic dimension. More recently constructed obstructions include separation profiles and certain versions of L^p -cohomology [BST12, Pan16].

Our main result is as follows:

Theorem 1.1. *Let G be a group admitting exponentially many fat bigons. Then G does not coarsely embed into any hyperbolic graph of bounded degree.*

Since groups which are hyperbolic relative to virtually nilpotent subgroups coarsely embed into hyperbolic graphs of bounded degree [DY05], we can also deduce that no group admitting exponentially many fat bigons is a subgroup of such a relatively hyperbolic group.

The exact definition of admitting exponentially many fat bigons is given in §2. Here we will just focus on examples, our main source of which is the following proposition.

Proposition 1.2. *(Proposition 3.1) Any finitely generated group with exponential growth and linear divergence admits exponentially many fat bigons.*

We recall the definition of divergence in Section 3. Examples of groups with linear divergence include direct products of infinite groups, groups with infinite center, groups satisfying a law (e.g., solvable groups) [DS05], all uniform [KL97] and many non-uniform [DMS10, CDG10, LB15] higher-rank lattices.

Corollary 1.3. *Let G be a virtually solvable finitely generated group. Then G coarsely embeds in some hyperbolic group if and only if G is virtually nilpotent.*

Proof. By Assouad’s Theorem [Ass82], every virtually nilpotent group can be coarsely embedded into some \mathbb{R}^n , and \mathbb{R}^n embeds into \mathbb{H}^{n+1} (as a horosphere) and hence into some hyperbolic group.

If G is not virtually nilpotent, then it has exponential growth [Mil68, Wol68]. Also, it has linear divergence ([DS05, Corollary 6.9] and [DMS10, Proposition 1.1]) and hence it cannot embed into any hyperbolic group by Proposition 1.2 and Theorem 1.1. \square

Corollary 1.4. *Let $m, n \in \mathbb{Z}$ with $|m| \leq |n|$. Then $BS(m, n)$ coarsely embeds into a hyperbolic group if and only if $m = 0$ or $|n| \leq 1$.*

Proof. If $m = 0$ or $|m| = |n| = 1$ then $BS(m, n)$ is virtually free or virtually abelian, so either is a hyperbolic group or coarsely embeds into one.

If $|m| = 1$ and $|n| > 1$ then $BS(m, n)$ is solvable with exponential growth so does not coarsely embed in a hyperbolic space by Corollary 1.3. When $1 < |m| < |n|$, $BS(1, 2)$ coarsely embeds into $BS(m, n)$, since $BS(1, 2)$ is

isomorphic to a subgroup of $BS(2, 4)$ which is quasi-isometric to $BS(m, n)$. It remains to check the case $1 < |m| = |n|$. In this case $BS(m, n)$ has a finite index subgroup isomorphic to $\mathbb{Z} \times F_n$ which has exponential growth and linear divergence, so we are done by Proposition 1.2. \square

The technique behind the proof of Proposition 1.2 yields more restrictions for groups with exponential growth.

Corollary 1.5. *Let G be a finitely generated group with exponential growth. If there exists a non-principal ultrafilter ω , a scaling sequence $(d_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ and a constant C such that*

- $d_n \leq C d_{n-1}$ for infinitely many n ,
- the asymptotic cone $\text{Con}^\omega(G, (d_n))$ has no cut points,

then G admits exponentially many fat bigons.

Proof. By [OOS09, Theorem 6.1], the group G has linear divergence on an unbounded subsequence. The proof of Proposition 1.2 will show that this suffices to deduce that G admits exponentially many fat bigons. \square

In particular, the lacunary hyperbolic groups with “slow non-linear divergence” constructed in [OOS09] do not coarsely embed into any hyperbolic group, and in particular they are not subgroups of any hyperbolic group.

Finally, in Proposition 3.2, we give a criterion for a $C'(1/6)$ small cancellation group to have exponentially many fat bigons. This can be used to give an explicit example of a small cancellation group that does not coarsely embed in, and in particular is not a subgroup of any hyperbolic group. This is in contrast with the $C(6)$ small cancellation subgroups of hyperbolic groups constructed by Kapovich-Wise [KW01].

We finish with two natural questions.

Question 1.6. Which (infinitely presented) small cancellation groups admit a coarse embedding into some hyperbolic group? Which are subgroups of some hyperbolic group?

Question 1.7. Is there an elementary amenable group with exponential growth, or a group of intermediate growth that admits a coarse embedding into some hyperbolic group?

Acknowledgements. The authors were supported in part by the National Science Foundation under Grant No. DMS-1440140 at the Mathematical Sciences Research Institute in Berkeley during Fall 2016 program in Geometric Group Theory. The authors would also like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme “Non-Positive Curvature, Group Actions and Cohomology” where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

2. FAT BIGONS

Given a metric space (X, d) $r > 0$ and $x \in X$ we denote by $B_r(x)$ the closed ball of radius r centred at x , and given a subset Y of X we denote the closed r -neighborhood of Y in X by $[Y]_r = \{x \in X \mid d(x, Y) \leq r\}$.

Definition 2.1. Let X be a metric graph, with base vertex x_0 , and let x be a vertex. Given $L, s, C \geq 0$, an (L, s, C) -bigon at x is given by two walks α_1, α_2 from x_0 to x with the following properties:

- (1) $l(\alpha_i) \leq Ld(x_0, x)$,
- (2) for $B = [\{x_0, x\}]_C$, we have $d(\alpha_1 - B, \alpha_2 - B) > s$.

Denote by $\mathcal{B}(L, s, C)$ the set of vertices x so that there exists an (L, s, C) -bigon at x .

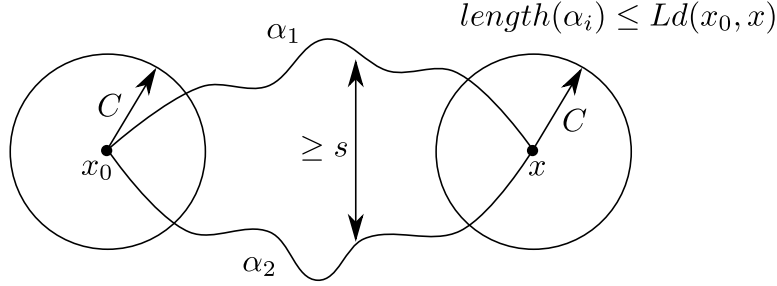


Figure 1. An (L, s, C) -bigon at x . The two paths connect the basepoint x_0 to some x , stay far from each other in the middle and are not too long.

Definition 2.2. Let X be a graph, with basepoint x_0 . We say that X has *exponentially many fat bigons* if there exist constants $c, L > 1$ such that for every s there exists a C so that the the function $g(n) = |\mathcal{B}(L, s, C) \cap B_n(x_0)|$ is bounded from below by c^n for infinitely many $n \in \mathbb{N}$.

We say that X has *no fat bigons* if for every L there exists s so that for every C we have that $\mathcal{B}(L, s, C)$ is a bounded subset of X .

Having no fat bigons is a strong negation of having exponentially many bigons. Our goal for this section is the following:

Theorem 2.3. *Let X be a graph with exponentially many fat bigons. Then X does not coarsely embed into any hyperbolic graph of bounded geometry.*

For notational purposes let us recall the definition of a coarse embedding. Given two graphs X, Y , with vertex sets VX, VY respectively, a *coarse embedding* is a map $f : VX \rightarrow VY$, a constant $K \geq 1$ and a function $\rho_- : \mathbb{N} \rightarrow \mathbb{N}$ such that $\rho(n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$(1) \quad \rho(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq Kd_X(x, y).$$

The proof of Theorem 2.3 is given as a pair of lemmas.

Lemma 2.4. *Let X, Y be bounded degree graphs. If X coarsely embeds into Y and X has exponentially many fat bigons, then so does Y .*

The idea of proof is that, despite the fact that the distance from the basepoint of a point of X could decrease drastically after applying a coarse embedding, this cannot happen for too many points because the growth of Y is (at most) exponential. More specifically, there must be many points x so that there is a fat bigon at x and the distance from the basepoint of Y to $f(x)$ is linear in the distance from the basepoint of X to x . For such x , there is a fat bigon at $f(x)$ (with slightly worse constants).

Proof. Let f be a coarse embedding of X into Y and let K, ρ satisfy (1). Increasing K and subtracting an additive constant from ρ we may assume that $f(x_0) = y_0$. Fix r such that $\rho(r) > 0$ and let Δ_Y be the maximal vertex degree of Y . By assumption there exist constants $d, L > 1$ such that for all $s \geq 0$ there is a constant C such that

$$|\mathcal{B}(L, s, C) \cap B_n(x_0)| \geq d^n$$

holds for all n in an infinite subset $I \subseteq \mathbb{N}$, while $|B_n(y_0) \cap VY|$ grows (at most) exponentially fast in n . Setting $c = \frac{1}{2}(1+d) > 1$, a simple calculation shows that $\mathcal{A}_\epsilon = \{x \in \mathcal{B}(L, s, C) : d(y_0, f(x)) > \epsilon d(x_0, x)\}$ has the property that

$$|\mathcal{A}_\epsilon \cap B_n(x_0)| \geq c^n$$

whenever $\epsilon < \frac{1}{2}(\log_c(\Delta_Y + 1))^{-1}$ and $n \in I$ is at least $2r/\epsilon$. Fix such an $\epsilon > 0$ and set $\mathcal{A} = \mathcal{A}_\epsilon$.

Claim. $f(\mathcal{A} \cap B_n(x_0)) \subset \mathcal{B}(KL\epsilon^{-1}, \rho(s) - 2K, KC + K) \cap B_{Kd(x_0, x)}(y_0)$.

Proof of Claim. Let $x \in \mathcal{A}$. Since f is K -Lipschitz and $f(x_0) = f(y_0)$, we have $f(x) \in B_{Kd(x_0, x)}(y_0)$.

If α_1, α_2 form a (L, s, C) -bigon at x , we can apply f to the vertices of the α_i and connect consecutive points by geodesics in Y , thereby obtaining new walks α'_1, α'_2 from y_0 to $f(x)$. The length of α'_i is at most K times the length of α_i , and hence $|\alpha'_i| \leq KL\epsilon^{-1}d_Y(y_0, f(x))$.

Given two vertices $v'_1 \in \alpha'_1, v'_2 \in \alpha'_2$ not in $[\{y_0, f(x)\}]_{KC+K}$ there are vertices $v_i \in \alpha'_i$ and $w_i \in \alpha_i$ such that $d_Y(v_i, v'_i) \leq K, f(w_i) = v_i$ and $w_i \notin [\{x_0, x\}]_C$ for $i = 1, 2$. Hence $d_X(w_1, w_2) > s$, so $d_Y(v_1, v_2) \geq \rho(s)$ by assumption and $d_Y(v'_1, v'_2) \geq \rho(s) - 2K$, as required. \square

Since $|f^{-1}(v)| \leq (\Delta_Y + 1)^r$ for each $v \in VY$, we see that

$$|\mathcal{B}(LK\epsilon^{-1}, \rho(s) - K, KC + K) \cap B_{Kn}(y_0)| \geq (\Delta_Y + 1)^{-r} c^n$$

holds for all $n \in I$ greater than $2r/\epsilon$. This easily implies that Y has exponentially many fat bigons. \square

Lemma 2.5. *Let X be a δ -hyperbolic graph. Then X has no fat bigons.*

The idea of proof is the following. Suppose we have two paths connecting the same pair of points which stay far from each other in the middle. Then any point on a geodesic connecting the endpoints can be close to at most one of the paths. Hence, one of the two paths stays far from at least “half” of the geodesic. Traveling far from a geodesic in a hyperbolic space is expensive, hence the path that stays far from half of the geodesic is long. More precisely, we consider disjoint balls along the geodesic, and count how many are avoided by each path.

Proof. We start with an easy fact about hyperbolic spaces.

Claim 1. There exist $\epsilon, s_0 > 0$ so that for each $s \geq s_0$ the following holds. Let $x, y \in X$ and let B_1, \dots, B_k be disjoint balls of radius s centred on a geodesic $[x, y]$. Let α be a walk from x to y that avoids all B_i . Then $l(\alpha) \geq k \cdot (1 + \epsilon)^s$.

Proof of Claim 1. Consider (clearly disjoint) maximal subpaths $\alpha_1, \dots, \alpha_k$ of α such that the closest point projection of the endpoints of α_i onto $[x, y]$ is contained in B_i . For each i let β_i^- (respectively β_i^+) be a geodesic from the vertex preceding (resp. succeeding) α_i on α to a closest point on $[x, y]$ not contained in B_i . By a standard divergence argument (see e.g. [BH99, Proposition III.H.1.6]) $|\beta_i^-| + |\alpha_i| + |\beta_i^+| \geq (1 + \epsilon')^s$ for some uniform $\epsilon' > 0$. Closest point projections are 4δ -coarsely well-defined, so any geodesic connecting the end vertices of α_i has length at least $|\beta_i^-| + |\beta_i^+|$ (assuming s_0 is sufficiently large in comparison to δ). Combining these observations, we see that

$$|\alpha_i| \geq \frac{1}{2}(1 + \epsilon')^s \geq (1 + \epsilon)^s$$

whenever $\epsilon < \frac{\epsilon'}{2}$ and s is sufficiently large. \square

Fix L . The following claim implies that X has no fat bigons.

Claim 2. There exists s large enough so that for every C there exists n with the following property: Let $x, y \in X$ with $d(x, y) \geq n$. Let α_1, α_2 be paths from x to y so that $d(\alpha_1 - B, \alpha_2 - B) \geq s$, for $B = B_C(x_0) \cup B_C(x)$. Then either $l(\alpha_1) \geq Ld(x, y)$ or $l(\alpha_2) \geq Ld(x, y)$.

Proof of Claim 2. Fix s_0, ϵ as in Claim 1. Up to increasing s_0 , we can assume that for every C there exists $n = n(C)$ so that $t \geq n$ implies $k(t)(1 + \epsilon)^{s_0} \geq Lt$, where $k(t) = \lfloor (t - 2C - 2s_0)/(4s_0) \rfloor$. Let $s = 2s_0$, fix any C and let $n = n(C)$.

Suppose $d(x, y) \geq n$. We can find $2k(d(x, y))$ disjoint balls B_i of radius s_0 whose centres lie on $[x, y]$ at distance at least $C + s_0$ from the endpoints. At most one of α_1, α_2 can intersect any given B_i and hence, up to swapping indices we can assume that α_1 avoids at least $k(d(x, y))$ of the B_i . By Claim 1, the length of α_1 is at least $Ld(x, y)$, as required. \square

Claim 2 clearly implies that X has no fat bigons. \square

3. GROUPS WITH FAT BIGONS

Let X be a geodesic metric space. Following [DMS10] we define the divergence of a pair of points $a, b \in X$ relative to a point $c \notin \{a, b\}$ is the length of the shortest path from a to b avoiding a ball around c of radius $\frac{1}{2}d(c, \{a, b\}) - 2$. If no such path exists, then we define the divergence to be infinity. The divergence of a pair a, b , $Div(a, b)$ is the supremum of the divergences of a, b relative to all $c \in X \setminus \{a, b\}$.

The divergence of X is given by $Div_X(n) = \max \{Div(a, b) \mid d(a, b) \leq n\}$.

3.1. Linear divergence groups.

Proposition 3.1. *Let X be a Cayley graph of a infinite group. If $Div_X(n) \leq Dn$, then for every $x \in X$ with $d(1, x) = \lfloor \frac{n}{20} \rfloor$ and every $s \geq 0$ there exists a $(100D, s, 2s)$ -bigon at x .*

Proof. We need a simple lemma about the geometry of Cayley graphs of infinite groups first.

Claim. For any s the following holds. Let $[p, q]$ be a geodesic in X . Then there exists a geodesic ray β starting at p so that for each $w \in \beta$ either $d(w, p) \leq 2s$ or $d(w, [p, q]) > s$.

Proof. There exists a bi-infinite geodesic γ through p . We claim that we can choose β to be one of the two rays starting at p and contained in γ . If not there exist $w_1, w_2 \in \gamma$ on opposite sides of p so that $\ell_i = d(w_i, p) > 2s$ but $d(w_i, x_i) \leq s$ for some $x_i \in [p, q]$. Without loss, we assume $\ell_1 \leq \ell_2$.

Hence, $\ell_1 + \ell_2 = d(w_1, w_2) \leq 2s + d(x_1, x_2) \leq 2s + (\ell_2 - \ell_1) + 2s$, from which we deduce $\ell_1 \leq 2s$, a contradiction. \square

Let $x \in VX$ and $n \in \mathbb{N}$ satisfy $Div_X(n) \leq Dn$ and $d(1, x) = \lfloor \frac{n}{20} \rfloor$; let us construct a $(100D, s, 2s)$ -bigon at x . If $d(1, x) \leq 4s$ there is nothing to prove. Let α_1 be any geodesic from 1 to x . Using the claim, let β, β' be geodesic rays starting at 1 and x respectively so that for every w on either β or β' at distance larger than $2s$ from the starting point we have $d(w, [1, x]) > s$. We can form α_2 by concatenating

- a sub-geodesic of β of length $10d(1, x)$, from 1 to a vertex y ,
- a path of length at most Dn that avoids $N_s([1, x])$ (whose existence is guaranteed by linear divergence) connecting y to a vertex $y' \in \beta'$
- a sub-geodesic of β' from y' to x .

It is easily seen that we have described a $(100D, s, 2s)$ -bigon at x . \square

3.2. More fat bigons. Relations in a $C'(\frac{1}{6})$ small cancellation group define isometrically embedded cyclic subgraphs in the appropriate Cayley graph (cf [LS01, Gro03]), so are natural examples of fat bigons. Therefore we obviously have the following:

Proposition 3.2. *Let G be a group which admits a $C'(\frac{1}{6})$ small cancellation presentation $G = \langle S \mid R \rangle$, where each $r \in R$ is cyclically reduced and no word in R can be obtained from any other via cyclic conjugation and/or inversion.*

If there are constants $c > 1$, $C \geq 0$ and an infinite subset $I \subseteq \mathbb{N}$ such that for each $n \in I$, $|\{r \in R \mid n - C \leq |r| \leq n + C\}| > c^n$, then $X = \text{Cay}(G, S)$ admits exponentially many fat bigons.

One way to build such a collection of relations is as follows. Set $S = \{a, b, c\}$. For each non-trivial word $w = \{a, b\}$, define $r_w = cwc^2w \dots c^{12}w$. The collection $R = \{r_w \mid w \in F(a, b)\}$ satisfies the hypotheses of the above proposition with $c = 3$, $C = 0$ and $I = \{12n + 78 \mid n \in \mathbb{N}\}$. If desired, we can ensure the group we construct is lacunary hyperbolic by instead taking $R = \{r_w \mid w \in F(a, b), |w| \in I\}$ for some suitably sparse infinite subset $I \subseteq \mathbb{N}$.

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MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, WOODSTOCK ROAD, OXFORD OX2 6GG, UK

E-mail address: `david.hume@maths.ox.ac.uk`

DEPARTMENT OF MATHEMATICS, ETH ZÜRICH, 8092 ZÜRICH, SWITZERLAND

E-mail address: `sisto@math.ethz.ch`